Geometry of horospherical products.

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Motivations

Theorem (Farb-Mosher, 1999)
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Theorem (Eskin-Fisher-Whyte, 2012)
Classification up to quasi-isometry of Diestel-Leader graphs $\mathrm{DL}(\mathrm{p}, \mathrm{q})$ and of solvable Lie groups $\mathrm{Sol}(\mathrm{p}, \mathrm{q})$.

Theorem (Eskin-Fisher-Whyte, 2012)
There exists a regular graph which possess an isometry group acting transitively on it which is not quasi-isometric to any Cayley graph.

Gromov hyperbolic, Busemann spaces

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Examples :

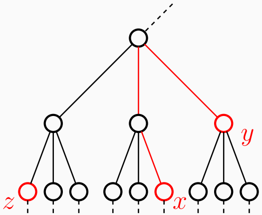
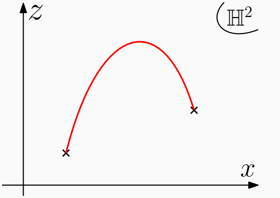


Figure 1 – Tree



$\mathbb{R}^2; ds^2 = e^{-2z}dx^2 + dz^2$

Figure 2 – Log model of the hyperbolic plane

Busemann functions or "height"

Definition (Gromov boundary and height function)
Let $\delta \geq 0$ and (X, d_X) be a δ-hyperbolic space and let $x_0 \in X$.

$$\partial_{x_0} X := \{\text{Geodesic rays starting at } x_0\} / \sim$$

Let $a \in \partial_{x_0} X$, $k \in a$. The **height function** h_X on X in regards to a is :

$$\forall x \in X,\ h_X(x) = -\beta_a(x) = \limsup_{t \to +\infty} \big(d_X(x, k(t)) - t \big)$$

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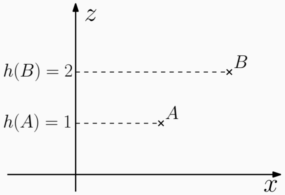

Figure 3 – Height in $\mathbb{H}^2$

$h(x, z) = z$ (in the Log model)

Vertical geodesics

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Let (X, d_X) be a δ-hyperbolic space. We fix $a \in \partial X$. A geodesic line is called **vertical** if one of its half-line is equivalent to a ray in a.

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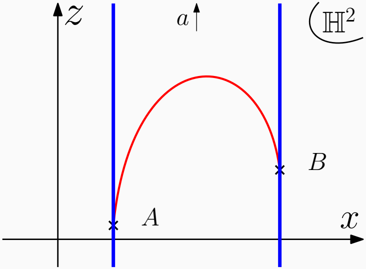

Figure 4 – Vertical geodesics of $\mathbb{H}^2$

Horospherical products

Definition and Examples

Definition (Horospherical product)
Let X and Y be two δ-hyperbolic spaces. Let h_X and h_Y be their respective height functions. The **horospherical product** $X \bowtie Y$ is :

$$X \bowtie Y := \left\{(x,y) \in X \times Y \mid h_X(x) = -h_Y(y)\right\} \left(= \bigcup_{z \in \mathbb{R}} X_z \times Y_{-z} \right)$$

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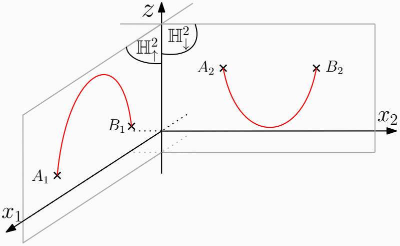

Figure 5 – $\mathbb{H}^2 \bowtie \mathbb{H}^2 = \mathrm{Sol}$

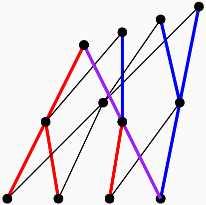

Figure 6 – $T_3 \bowtie T_3 = \mathrm{Cay}(\mathbb{Z}_2 \wr \mathbb{Z})$

Distance on $X \bowtie Y$

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The distance $d_{X \bowtie Y}$ is the length path metric induced by $\frac{d_X + d_Y}{2}$ on $X \times Y$.

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Theorem A (F, 2020)

$$d_{X \bowtie Y} = d_X + d_Y - \Delta h \pm C$$

Geodesic segments

Corollary
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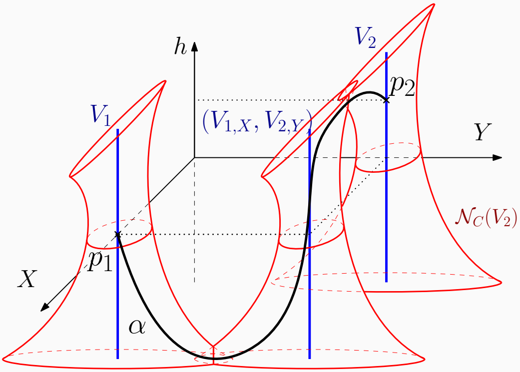

Figure 7 – Geodesic segment of $X \bowtie Y$

Geodesic lines of $X \bowtie Y$

Theorem B (F, 2020)
Any geodesic line α of $X \bowtie Y$ verifies at least one of the two following statements :

1. α is close to a X-type geodesic.
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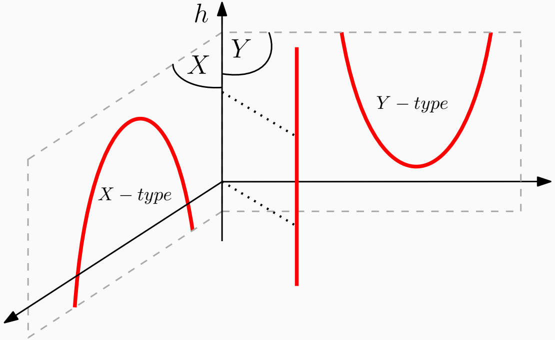

Figure 8 – Geodesic line types

Visual boundary of $X \bowtie Y$

Corollary
Let $a_X \in \partial X$ et $a_Y \in \partial Y$. The visual boundary of $X \bowtie Y$ is :

$$\partial(X \bowtie Y) = \Big((\partial X \setminus \{a_X\}) \times \{a_Y\} \Big) \bigcup \Big((\partial Y \setminus \{a_Y\}) \times \{a_X\} \Big)$$

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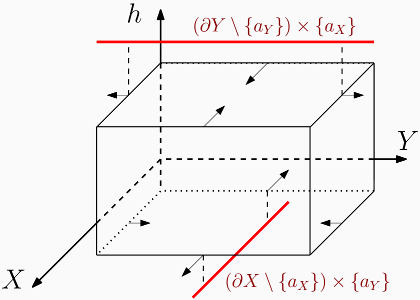

Figure 9 – Visual boundary of $X \bowtie Y$

Geometric rigidity of self quasi-isometries

Geometric rigidity

Assume :

- X and Y are endowed with *admissible desintegrable measures*.
- X and Y do not share the same parameter of *exponential divergence* ($m \neq n$).

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Theorem C : Geometric rigidity (F, 2022)
Let $\Phi : X \bowtie Y \to X \bowtie Y$ be a (k, c)-quasi-isometry. There exist $C(k, c, \bowtie) \in \mathbb{R}$, $\Phi^X : X \to X$ and $\Phi^Y : Y \to Y$ two quasi-isometries such that :

$$d_{\bowtie}\left(\Phi, (\Phi^X, \Phi^Y)\right) \leq C$$

Geometric rigidity consequences

Theorem D (F, 2022)
Let $S_1 = \mathbb{R} \ltimes_{A_1} N_1$ and $S_2 = \mathbb{R} \ltimes_{A_2} N_2$ be two simply connected, negatively curved Lie groups (Heintze groups) such that $\mathrm{tr}(A_1) \neq \mathrm{tr}(A_2)$, then :

$$\mathrm{QI}\left(\mathbb{R} \ltimes_{\mathrm{Diag}(A_1, -A_2)} (N_1 \times N_2)\right) = \mathrm{Bilip}(N_1) \times \mathrm{Bilip}(N_2)$$

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Bi-Lipschitz for the left-invariant Hamenstädt distance $\forall n, n' \in N_i$:

$$d_{\mathrm{Ham}}(n, n') = \exp\left(-\frac{1}{2} \lim_{s \to +\infty} \left(2s - d_{S_i}\left((-s, n), (-s, n')\right)\right)\right)$$

Admissible desintegrable measures

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(a) $\forall z \in \mathbb{R},\ \exists \mu_z^X$ measure on X_z such that $\forall U \subset X$:

$$\mu^X(U) = \int_{z \in \mathbb{R}} \mu_z^X(U_z) \mathrm{d}z$$

Where :

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(c) There exists $m > 0$ such that $\forall z_0 \in \mathbb{R}$, $\forall U \subset X_{z_0}$:

$$\forall z \leq z_0,\ e^{m(z_0 - z)}\mu_{z_0}^X(U) \asymp \mu_z^X\left(\pi_z(U)\right)$$

Where :

- $U_z = U \cap h^{-1}(z)$
- $\pi_z(U)$ is the *vertical* projection of U on X_z.

Box-tiling of $X \bowtie Y$

Definition : Box $\mathcal{B}(x,R)$ in X.
Let $x \in X$, $R > 0$ and $\mathcal{C}(x)$ a cell of nucleus x. ($\mathcal{C}(x) \sim$ rough horizontal disk centered at x)

$$\mathcal{B}(x,R) := \bigcup_{z \in]h(x)-R; h(x)]} \pi_z(\mathcal{C}(x))$$

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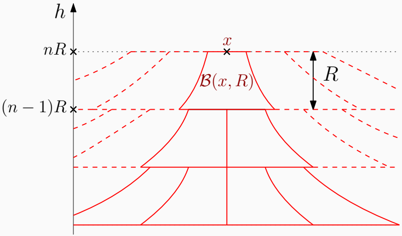

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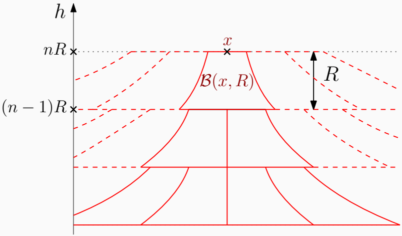

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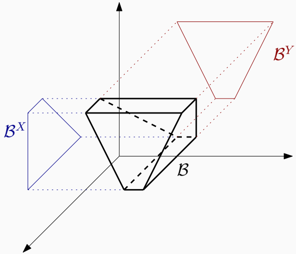

Figure 11 – Box in $X \bowtie Y$

Proof of Theorem C

Tile $X \bowtie Y$ with boxes $\mathcal{B}(R)$ of scale R ($\sim$ cubes of side R)

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2) **Vertical quadrilaterals** + good scale $R \Rightarrow$ On almost all boxes $\mathcal{B}(R)$ at scale R, $\Phi_{|\mathcal{B}} \approx \left(\Phi^X_{\mathcal{B}}, \Phi^Y_{\mathcal{B}}\right)$

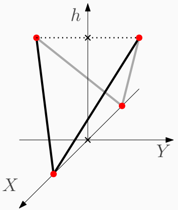

Figure 12 – Vertical quadrilateral

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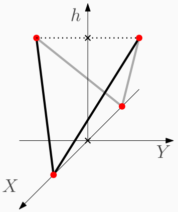

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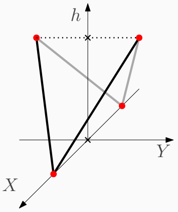
4) Sequence of growing boxes $\mathcal{B}(L_n) \Rightarrow \Phi \approx \left(\Phi^X, \Phi^Y\right)$



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Theorem C $\Rightarrow$ Theorem D

We have : $(\mathbb{R} \ltimes_{A_1} N_1) \bowtie (\mathbb{R} \ltimes_{A_2} N_2) = \mathbb{R} \ltimes_{\mathrm{Diag}(A_1, -A_2)} (N_1, N_2)$

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4) $\Phi \approx (\mathrm{id}_{\mathbb{R}}, \Psi_1, \Psi_2)$ $\quad \big(\Rightarrow (1, c)$-QI$\big)$

Thank you for your attention.