Geometry and quasi-isometry rigidity of horospherical products

Tom Ferragut

Differential Topology Seminar of Kyoto University

Hints on horospherical products

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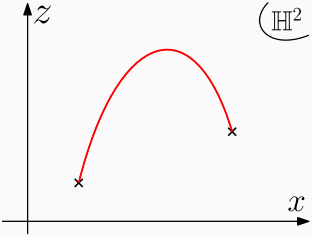

Figure 1 – Log model of the hyperbolic plane

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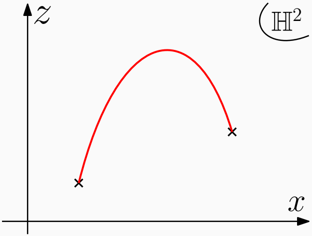

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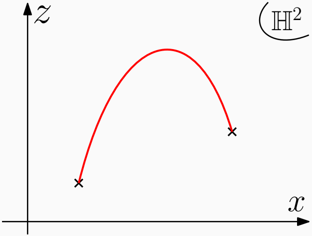

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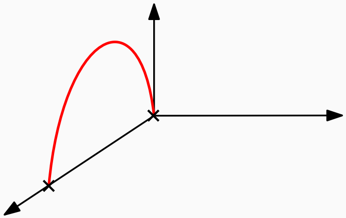

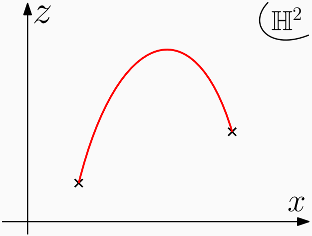

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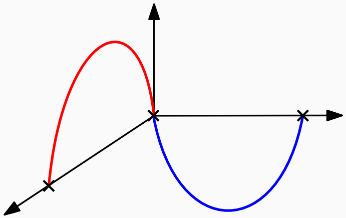

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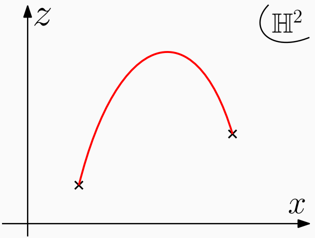

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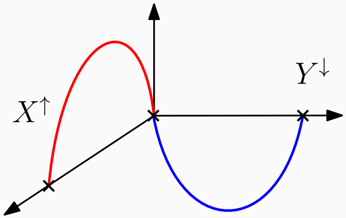

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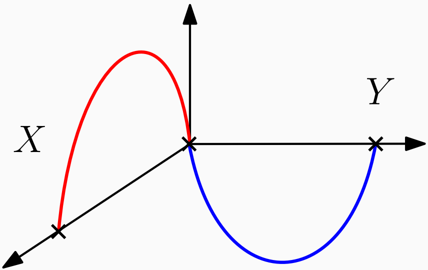

Figure 2 – Horospherical product $X \bowtie Y$

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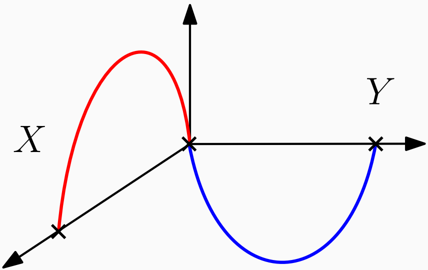

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Study the geometry of $X \bowtie Y$:

- Shapes and lengths of geodesics
- Description of the visual boundary
- Geometric rigidity of quasi-isometries

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Theorem (Eskin-Fisher-Whyte, 2012)
There exists a regular graph which possess an isometry group acting transitively on it which is not quasi-isometric to any Cayley graph.

Gromov hyperbolic and Busemann spaces

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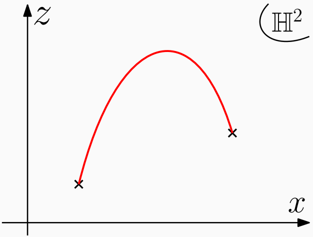

$$\mathbb{R}^2; ds^2 = e^{-2z}dx^2 + dz^2$$

Figure 3 – Log model of the hyperbolic plane

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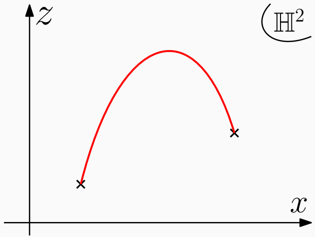
- Trees



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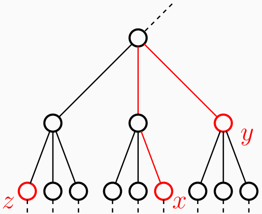

Figure 4 – Tree

Height functions and vertical geodesics

Definitions
Let $x_0 \in X$ and fix $a \in \partial_{x_0} X$. Let $k \in a$, the **height function** h_X on X with respect to a is :

$$\forall x \in X,\ h_X(x) = -\beta_a(x) = \sup_{t \to +\infty} \big(d_X(x, k(t)) - t \big)$$

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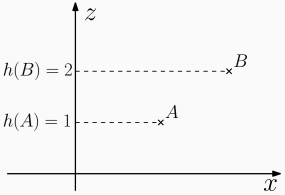

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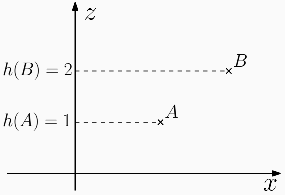

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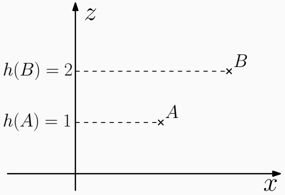

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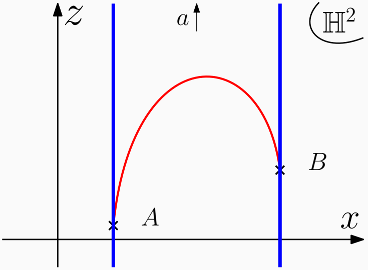

Figure 6 – Vertical geodesics of $\mathbb{H}^2$

Horospherical products and their geodesics

Definition (Horospherical product)
Let X and Y be two δ-hyperbolic spaces. Let h_X and h_Y be their respective height functions. The **horospherical product** $X \bowtie Y$ is :

$$X \bowtie Y := \left\{(x,y) \in X \times Y \mid h_X(x) = -h_Y(y)\right\} \left(= \bigcup_{z \in \mathbb{R}} X_z \times Y_{-z}\right)$$

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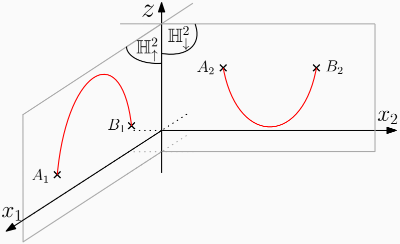

Figure 7 – $\mathbb{H}^2 \bowtie \mathbb{H}^2 = \mathrm{Sol}$

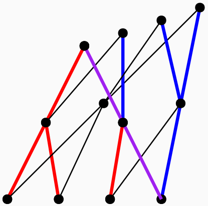

Figure 8 – $T_3 \bowtie T_3 = \mathrm{Cay}(\mathbb{Z}_2 \wr \mathbb{Z})$

Distance on $X \bowtie Y$

Definition ($d_{X \bowtie Y}$)
The distance $d_{X \bowtie Y}$ is the length path metric induced by $N(d_X, d_Y)$ on $X \times Y$, with N an *admissible* norm of $\mathbb{R}^2$.

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Theorem A (F, 2020)
There exists $C(\delta, N) \geq 0$ such that :

$$d_{X \bowtie Y} = d_X + d_Y - \Delta h \pm C$$

Sketch of proof for Theorem A

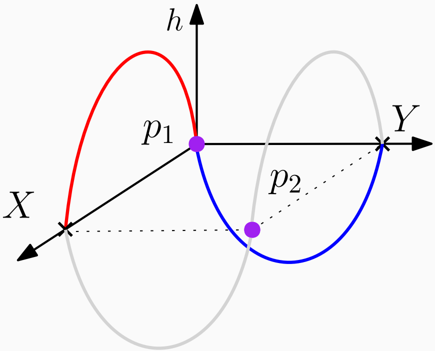

Figure 9 – Sketch

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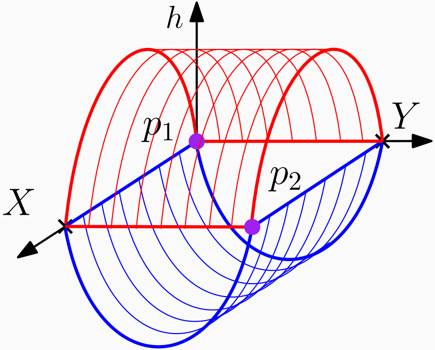

Figure 9 – Sketch

Length of geodesic projections

Proposition
Let X be a δ-hyperbolic space and let c be a path of X. If $[p,q]$ is a geodesic segment linking the endpoints of c, then for all $x \in [p,q]$ we have

$$d_X(x, \mathrm{im(c)}) \leq \delta \left|\log_2 \mathrm{l(c)}\right| + 1$$

If $h^+(c) \leq h^+\big([p,q]\big) - \Delta H$, Then :

$$l(c) \geq d_X(p,q) + 2^{\Delta H}$$

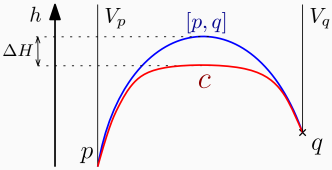

Figure 10 – A not high enough path

Distance description

Let $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$ be two points of $X \bowtie Y$ and let c be a path of $X \bowtie Y$ linking p_1 to p_2. Then

$$\begin{aligned} l(c) &\geq \frac{l(c_X) + l(c_Y)}{2} \\ &\geq d_X(x_1, x_2) + d_Y(y_1, y_2) - |h(p_1) - h(p_2)| + 2^{\Delta H_X} + 2^{\Delta H_Y} \end{aligned}$$

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Theorem A ($d_{X \bowtie Y}$)
There exists $C(\delta, N)$ such that

$$d_{X \bowtie Y} = d_X + d_Y - \Delta h \pm C$$

Corollary
$\left(X \bowtie Y, d_{l_1}\right)$ and $\left(X \bowtie Y, d_{l_2}\right)$ are $(1, C)$-quasi-isometric.

Geodesic segments

Corollary
A geodesic segment α is close to the union of three vertical geodesics.

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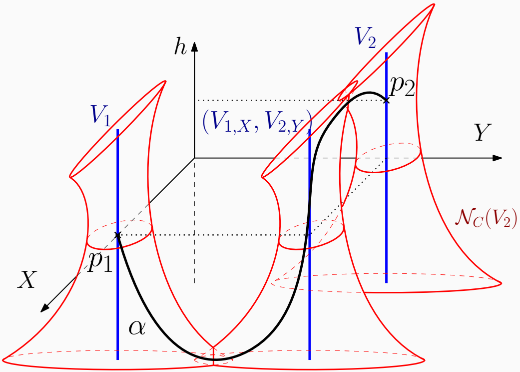

Figure 11 – Geodesic segment of $X \bowtie Y$

Geodesic lines of $X \bowtie Y$

Theorem B (F, 2020)
Any geodesic line α of $X \bowtie Y$ verifies at least one of the two following statements :

1. α is close to a X-type geodesic.
2. α is close to a Y-type geodesic.

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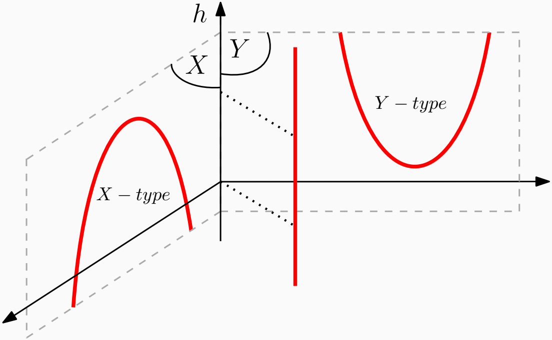

Figure 12 – Geodesic line types

Visual boundary of $X \bowtie Y$

Corollary
Let $a_X \in \partial X$ et $a_Y \in \partial Y$. The visual boundary of $X \bowtie Y$ is :

$$\partial(X \bowtie Y) = \Big(\big(\partial X \setminus \{a_X\}\big) \times \{a_Y\} \Big) \bigcup \Big(\{a_X\} \times \big(\partial Y \setminus \{a_Y\}\big) \Big)$$

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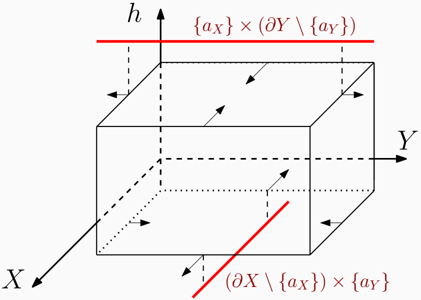

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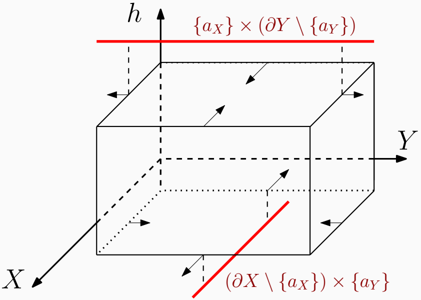

Figure 13 – Visual boundary of $X \bowtie Y$

Example : $\partial\big(\mathbb{R} \ltimes_{\mathrm{Diag}(A_1,-A_2)} (N_1 \times N_2)\big) = N_1 \times N_2$

Proof of Theorems B

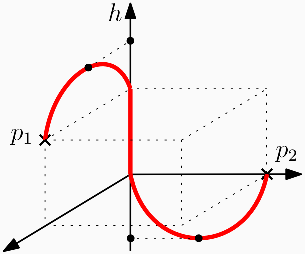

Figure 14 – Geodesic segment

- Two behaviours for geodesic segments : $\nearrow\searrow\nearrow$ or $\searrow\nearrow\searrow$.

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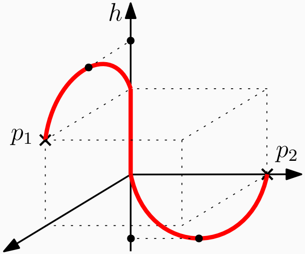

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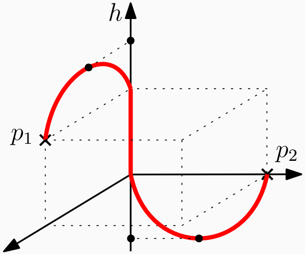

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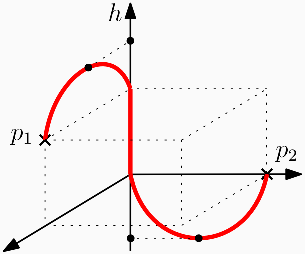

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- Geodesic ray classification $\Rightarrow$ Characterisation of the visual boundary of $X \bowtie Y$

Geometric rigidity of quasi-isometries

Geometric rigidity

Assume :

- X, Y, X' and Y' are endowed with *admissible disintegrable measures*.
- X and Y (resp. X' and Y') do not share the same *exponential growth parameter* $m > n$ (resp. $m' > n'$).

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Theorem C : Geometric rigidity (F, 2022)
Let $\Phi : X \bowtie Y \to X' \bowtie Y'$ be a (k, c)-quasi-isometry. There exist $C(k, c, \bowtie) \in \mathbb{R}$, $\Phi^X : X \to X'$ and $\Phi^Y : Y \to Y'$ two quasi-isometries such that :

$$d_{\bowtie}\left(\Phi, (\Phi^X, \Phi^Y)\right) \leq C$$

Geometric rigidity consequences

Theorem D (F, 2022)
Let $S_1 = \mathbb{R} \ltimes_{A_1} N_1$ and $S_2 = \mathbb{R} \ltimes_{A_2} N_2$ be two simply connected, negatively curved Lie groups (Heintze groups) such that $\operatorname{tr}(A_1) \neq \operatorname{tr}(A_2)$, then :

$$\mathrm{QI}\left(\mathbb{R} \ltimes_{\mathrm{Diag}(A_1, -A_2)} (N_1 \times N_2)\right) = \mathrm{Bilip}(N_1) \times \mathrm{Bilip}(N_2)$$

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Bi-Lipschitz for the left-invariant Hamenstädt distance $\forall n, n' \in N_i$:

$$d_{\mathrm{Ham}}(n, n') = \exp\left(-\frac{1}{2} \lim_{s \to +\infty} \left(2s - d_{S_i}\left((-s, n), (-s, n')\right)\right)\right)$$

Admissible desintegrable measures

Let (X, μ^X) be a Gromov, Busemann measured metric space, μ^X is an admissible desintegrable measure if :

Admissible desintegrable measures

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Where :

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(c) There exists $m > 0$ such that $\forall z_0 \in \mathbb{R}$, $\forall U \subset X_{z_0}$:

$$\forall z \leq z_0,\ e^{m(z_0 - z)}\mu^X_{z_0}(U) \asymp \mu_z^X\left(\pi_z(U)\right)$$

Where :

- $U_z = U \cap h^{-1}(z)$
- $\pi_z(U)$ is the *vertical* projection of U on X_z.

Box-tiling of $X \bowtie Y$

Definition : Box $\mathcal{B}(x,R)$ in X.
Let $x \in X$, $R > 0$ and $\mathcal{C}(x)$ a cell of nucleus x. ($\mathcal{C}(x) \sim$ rough horizontal disk centered at x)

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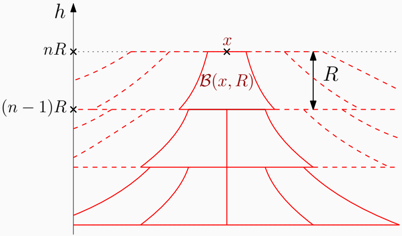

Figure 15 – Box Tiling of X

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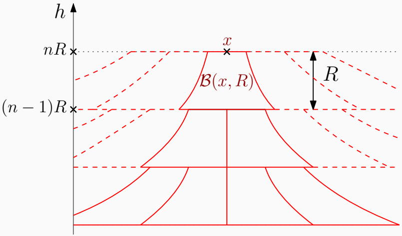

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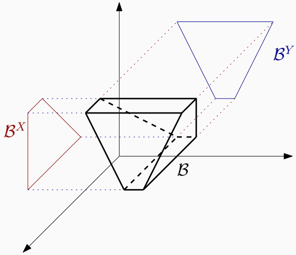

Figure 16 – Box in $X \bowtie Y$

Proof of Theorem C

Tile $X \bowtie Y$ with boxes $\mathcal{B}(R)$ of scale R.

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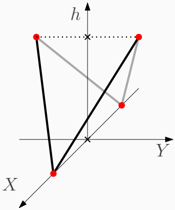

Figure 17 – Vertical quadrilateral

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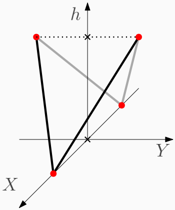

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4) Given two points sharing their height, Φ send them almost on the same height $\Rightarrow \Phi \approx \left(\Phi^X, \Phi^Y\right)$

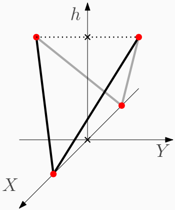

Figure 17 – Vertical quadrilateral

Theorem C $\Rightarrow$ Theorem D

We have : $(\mathbb{R} \ltimes_{A_1} N_1) \bowtie (\mathbb{R} \ltimes_{A_2} N_2) = \mathbb{R} \ltimes_{\mathrm{Diag}(A_1, -A_2)} (N_1 \times N_2)$

We want : $\mathrm{QI}\left(\mathbb{R} \ltimes_{\mathrm{Diag}(A_1, -A_2)} (N_1 \times N_2)\right) = \mathrm{Bilip}(N_1) \times \mathrm{Bilip}(N_2)$

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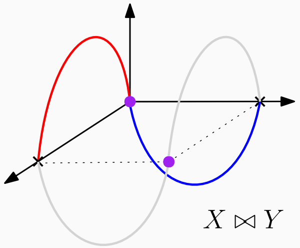
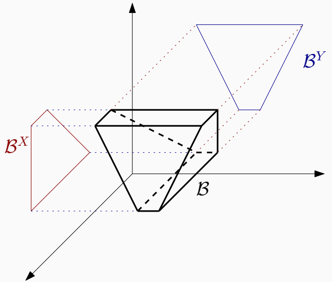
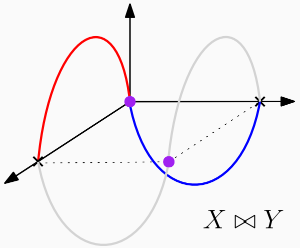
Perspectives

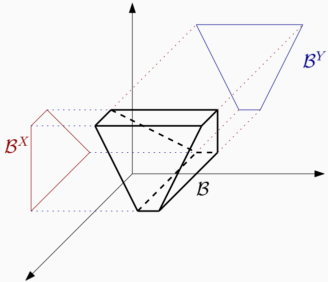




- Q.I. classification of $\mathbb{R} \ltimes (N_1 \times N_2)$.

Perspectives





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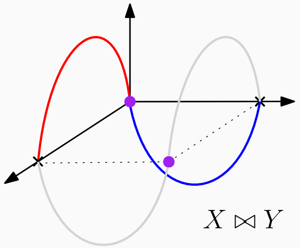
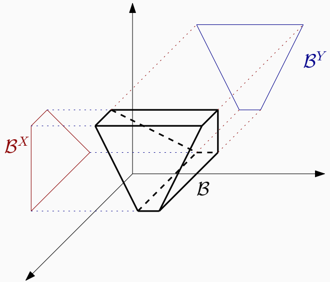
Perspectives





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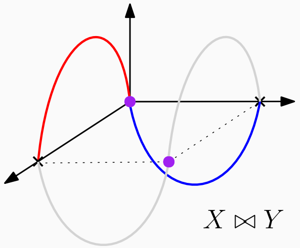

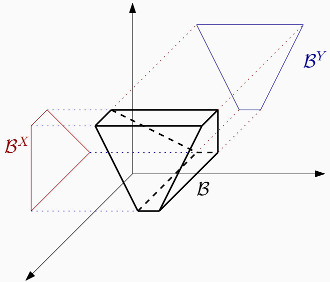

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- Use Irene Peng's methods to achieve similar results for the solvable Lie groups $\mathbb{R}^p \ltimes (N_1 \times N_2)$
- Remove some assumptions : $m \neq n$; (Busemann)
- What about $X_1 \bowtie X_2 \bowtie ... \bowtie X_n$?

Thank you for your attention.